Is there Dynamic Inefficiency when the rate of growth of GDP is greater than the safe real interest rate but lower than the expected average return on capital ? *

The question is technical but has great policy relevance. Another way to phrase it is to ask if it is plausible that, in the absense of Ricardian equivalence, but under the assumption of full employment, an increase in public debt would cause higher balanced growth welfare. I think that useful light on this question can be cast by a simple model with strong assumptions.

Throughout I will maintain some extreme simplifying assumptions. The first is that employment is exogenous: there is full employment & labor supply is exogenous. This note has nothing to do with Keynesian arguments about the effect of public debt if full employment is not guaranteed by assumption.

The second is that the economy consists of a state plus overlapping generations of agents who work when they are young and live off of their capital when they are old (standard Diamond model with productive capital). This means that there is not Ricardian equivalence and the timing of taxation affects the economy even given the public sector budget constraint. It also means that, in theory, the economy can be dynamically inefficient so that a lower constant capital to effective labor ratio causes a higher consumption to effective labor ratio.

The state will do only one thing. It issues safe one period inflation protected securities which are sued in period t and worth one unit of consumption good when they mature in period t+1. These payments are financed by the issuance of new securities and possibly by lump sum taxes. The state may also make lump sum transfer payments to the young or to the old.

For simplicity only assume that the population (and labor supply) are constant. The effective labor supply grows only because of exogenous labor augmenting technological progress. This just simplifies notation compared to a standard model with both population growth and technological progress.

It is assumed that each old person (of a continuum of measure 1 indexed by i) operates his or her own firm (these firms are liquidated after one period). Each firm has an idiosyncratic firm specific technology.

Output by firm i at time t is given by a standard Cobb Douglas production function.

$$Y_{it}=K_{it}^{\alpha}(A_{it}L_{it})^{1-\alpha} \quad (1)$$

Where

$$A_{it}=A_t(1+\epsilon_{it})^{1/(1-\alpha)} \quad (2)$$

Each firm has an idiosyncratic firm specific technology shock. ϵ_{it} has mean zero and variance $(\sigma)^2$.

So

$$A_{it}^{1-\alpha}=A_t^{1-\alpha}(1+\epsilon_{it}) \quad (3)$$

This shock is not verifiable (that is output is not verifiable) so each owner must bear all of the risk. It can't be born by outside shareholders or by the employee.

A_t grows geometrically

$$A_t=A_{t-1}(1+g) \quad (4)$$

Each old person i chooses L_{it} before ϵ_{it} is realized so L_{it} = 1 * and

$$Y_{it}=K_{it}^{\alpha}A_{it}^{1-\alpha} \quad (5)$$

I thank Nick Rowe for explaining to me that the assumption that employment is chosen before the shock is realized is a better motivation for the the result that all firms have the same number of employees than the odd assumption I originally made that each old person has the capacity to supervise no more than one young person.

In period t the state sells bonds for $B_t=bA_t$ units of consumption good. In period t+1 the bonds are redeemed for $bA_t(1+r_{t+1}t^s)$ where r_t^s is the safe rate of interest. $bA_{t+1}=(1+g)bA_t$ and the state distributes a pension of $bA_t(g-r_{t+1}^s)$ to the old.

Young agents divide their W_t income into consumption C_t^y , investment in their new firm K_{t+1} and bond purchases bA_t **.** They are identical and make identical choices so the subscript i is supressed.

$$C_t^y+bA_t+K_{t+1}=W_t \quad (6)$$

Young workers supply labor inelastically. The old decide how many young workers to hire (in equilibrium each hires one) promising pay W_t. Then uncertainty is resolved, production occurs, the old pay w_t and consume

$$C_{it}^o=bA_t+K_t+K_t^{\alpha}A_{it}^{1-\alpha}-W_t \quad (7)$$

There is no completely standard notation for the time subscript on consumption of the old. Here the t subscript refers to the time when the consumption occurs. This notational problem will not create difficulties because I will consider only balanced growth paths.

The expected return on capital paid in period t is $(1-\alpha)k_t^{\alpha-1}$ which must be greater than r_t^s .
If

$$(1-\alpha)k_t^{\alpha-1}>g \quad (8)$$

Then capital income is greater than investment, that is, total consumption is greater than total labor income. In models without risk, this is the condition for dynamic efficiency. It is however, consistent with dynamic inefficiency in this model. I assume that the inequaliy holds.

Assume agent i born in period t maximizes the expected value of

$$\ln\left(C_{it}^{y}\right)+\beta\ln\left(C_{it+1}^{o}\right) \quad (9)$$

This is the mathematically easiest of standard utility functions chosen for convenience.

The present value of the wage plus the discounted pension is

$$permanent\ income=W_t+\frac{bA_t\left(g-r_{t+1}^{s}\right)}{1+r_{t+1}^{s}} \quad (10)$$

Even for the case of logarithmic utility, there isn't a closed form solution for C_t^y as a function of w_t bA_t and the safe interest rate r_{t+1}^{s} . The only simple case occurs when r_{t+1}^{s} = g. then no matter what the risky return on capital is, a constant fraction of the wage

$$if\ g=r_{t+1}^{s}\ then\ C_t^y=\frac{W_t}{1+\beta} \quad (11)$$

In this special case, government bonds crowd out private investment one for one with

$$K_{t+1}=\frac{\beta W_t}{1+\beta}-bA_t \quad (12)$$

If r_{t+1}^{s} < g then government bonds crowd out private investment more than one for one as the expected pension payment causes higher consumption when young.

In this model the labor demand problem is a bit tricky, because the old don't know A_{it} when they chose to hire 1 young worker at wage W_t. At time t the old choose L_{it} (which will be equal to one in equilibrium) to maximize the expected value of utility

$$E\left[\ln\left(bA_t+K_t+K_t^{\alpha}\left(A_{it}L_{it}\right)^{1-\alpha}-W_tL_{it}\right)\right] \quad (13)$$

Equation (14) is a second order Taylor series approximation to (13)

with which implies the first order condition at L_{it} = 1

$$E\frac{(1-\alpha)K_t^{\alpha}A_{it}^{1-\alpha}-W_t}{K_t+bA_t+K_t^{\alpha}A_{it}^{1-\alpha}-W_t}=0 \quad (14)$$

Now define k = K/A and w = W/A and note that $A_{it}^{1-\alpha}/A_t^{1-\alpha}=1+\epsilon_{it}$

So the first order condition becomes

$$E\frac{(1-\alpha)k_t^{\alpha}(1+\epsilon_{it})-w_t}{k_t+b+k_t^{\alpha}(1+\epsilon_{it})-w_t}=0 \quad (15)$$

Which implies $w_t<(1-\alpha)k_t^{\alpha}$

The wage is lower than it would be in the absence of risk, because the old's marginal utility of consumption is higher when A_t is low. Another way of putting this is that the risk reduces labor demand, because higher employment causes the employer to bear more risk.

For small enough σ^2 one can use a second order Taylor series approximation to the utility function to obtain an approximate first order condition at at L_{it} = 1

$$\frac{(1-\alpha)k_t^{\alpha}-w_t}{k_t+b+k_t^{\alpha}-w_t}-\frac{2(1-\alpha)k_t^{\alpha}(k_t+b+k_t^{\alpha}-w_t)\sigma^2}{2(k_t+b+k_t^{\alpha}-w_t)^2}\approx 0 \quad (16)$$

Which immediately simplifies to

$$w_t\approx(1-\alpha)k_t^{\alpha}(1-\sigma^2) \quad (17)$$

Finally assume balanced growth, so the lower case variables are constant. If r^s = g balanced growth occurs iff

$$k(1+g)=\frac{K_{t+1}}{A_t}=k+\frac{\beta w}{1+\beta}-b \quad (18)$$

Recall that $\frac{\beta w}{1+\beta}-b<k+\frac{\beta(1-\alpha)k^{\alpha}}{1+\beta}-b \quad (19)$

If $r^s<g$ balanced growth k is reduced due to the effect of pensions on saving (this effect increases in b).

The welfare of an agent born at time t is

$$U_t=\ln\left(\frac{w}{(1+\beta)(1+g)}\right)+\beta E[\ln(b+k+k^{\alpha}(1+\epsilon_{it})-w)]+(t+1)\ln(1+g) \quad (20)$$

An increase in b has three effects in equilibrium. It causes a shift of saving from risky capital to safe bonds. For constant expected return on capital, If r^s and w, agents are indifferent about this shift. If If $r^s<g$, it causes an increase in pension payments which unambiguously cause higher welfare and it causes a reduction of w both by causing reduced k and by causing reduced expected consumption and increased absolute risk aversion of the old. The reduction in w implies a transfer from the young to the old with each agent receiving 1+g units of consumption when old for each unit of consumption lost when young. Since w is not stochastic, this transfer involves no risk. If $r^s<g$ this transfer causes higher welfare for each generation. Agents are indifferent about consuming a bit less when young, saving a bit more and receiving a safe return of r^s g so they strictly prefer the effect of reduced wages which are equivalent to making them save and receive a safe return of g.

This means that, if the safe rate of interest is less than the rate of growth, the economy is dynamically inefficient. This is the case even though total capital income is greater than investment, that is, total consumption is greater than labor income.

The usual simple national income and product accounts based formula for determining whether an economy is dynamically efficient due to Abel, Mankiw, Summers, and Zeckhauser does not work in this case.

* I want to thank Nick Rowe for a very useful comment.