Is there Dynamic Inefficiency when the rate of growth of GDP is greater than the safe real interest rate but lower than the expected average return on capital ? *

The question is technical but has great policy relevance. Another way to phrase it is to ask if it is plausible that, in the absence of Ricardian equivalence, but under the assumption of full employment, an increase in public debt would cause higher balanced growth welfare. I think that useful light on this question can be cast by a simple model with strong assumptions.

Throughout I will maintain some extreme simplifying assumptions. The first is that employment is exogenous: there is full employment & labor supply is exogenous. This note has nothing to do with Keynesian arguments about the effect of public debt if full employment is not guaranteed by assumption.

The second is that the economy consists of a state plus overlapping generations of agents who work when they are young and live off of their capital when they are old (standard Diamond model with productive capital). This means that there is not Ricardian equivalence and the timing of taxation affects the economy even given the public sector budget constraint. It also means that, in theory, the economy can be dynamically inefficient so that a lower constant capital to effective labor ratio causes a higher consumption to effective labor ratio.

The state will do only one thing. It issues safe one period inflation protected securities which are sued in period t and worth one unit of consumption good when they mature in period t+1. These payments are financed by the issuance of new securities and possibly by lump sum taxes. The state may also make lump sum transfer payments to the young or to the old.

For simplicity only assume that the population (and labor supply) are constant and normalized to 1. The effective labor supply grows only because of exogenous labor augmenting technological progress. This just simplifies notation compared to a standard model with both population growth and technological progress.

It is assumed that each old person (of a continuum of measure 1 indexed by i) operates his or her own firm (these firms are liquidated after one period). Each firm has an idiosyncratic firm specific technology.

Output by firm i at time t is given by a standard Cobb Douglas production function.

$$Y_{it} = K_{it}^{\alpha}(A_{it} L_{it})^{1-\alpha} \quad (1)$$

Where

$$A_{it} = A_t (1+\epsilon_{it})^{1/(1-\alpha)} \quad (2)$$

Each firm has an idiosyncratic firm specific technology shock. ϵ_{it} has mean zero and variance $(\sigma)^2$.

So

$A_{it}^{1-\alpha}=A_t^{1-\alpha}(1+\epsilon_{it})$ (3)

This shock is not verifiable (that is output is not verifiable) so each owner must bear all of the risk. It can't be born by outside shareholders or by the employee.

A_t grows geometrically

$A_t=A_{t-1}(1+g)$ (4)

Each old person i chooses L_{it} before ϵ_{it} is realized so L_{it} = 1 * and

$Y_{it}=K_{it}^{\alpha}A_{it}^{1-\alpha}$ (5)

I thank Nick Rowe for explaining to me that the assumption that employment is chosen before the shock is realized is a better motivation for the the result that all firms have the same number of employees than the odd assumption I originally made that each old person has the capacity to supervise no more than one young person.

In period t the state sells bonds for $B_t=bA_t$ units of consumption good. In period t+1 the bonds are redeemed for $bA_t(1+r_{t+1}^s)$ where r_t^s is the safe rate of interest. $bA_{t+1}=(1+g)bA_t$ and the state distributes a pension of $bA_t(g-r_{t+1}^s)$ to the old.

Young agents divide their W_t income into consumption C_t^y , investment in their new firm K_{t+1} and bond purchases bA_t . They are identical and make identical choices so the subscript i is supressed.

$C_t^y+bA_t+K_{t+1}=W_t$ (6)

Young workers supply labor inelastically. The old decide how many young workers to hire (in equilibrium each hires one) promising pay W_t. Then uncertainty is resolved, production occurs, the old pay w_t and consume

$C_{it}^o=bA_t+K_t+K_t^{\alpha}A_{it}^{1-\alpha}-W_t$ (7)

There is no completely standard notation for the time subscript on consumption of the old. Here the t subscript refers to the time when the consumption occurs. This notational problem will not create difficulties because I will consider only balanced growth paths.

The expected return on capital paid in period t is $(1-\alpha)k_t^{\alpha-1}$ which must be greater than r_t^s .
If
$(1-\alpha)k_t^{\alpha-1}>g$ (8)

Then capital income is greater than investment, that is, total consumption is greater than total labor income. In models without risk, this is the condition for dynamic efficiency. It is however, consistent with dynamic inefficiency in this model. I assume that the inequality holds.

Assume agent i born in period t maximizes the expected value of

$$\ln\left(C_{it}^{y}\right)+\beta\ln\left(C_{it+1}^{o}\right) \quad (9)$$

This is the mathematically easiest of standard utility functions chosen for convenience.

The present value of the wage plus the discounted pension is

$$permanent\,income=W_t+\frac{bA_t\left(g-r_{t+1}^{s}\right)}{1+r_{t+1}^{s}} \quad (10)$$

Even for the case of logarithmic utility, there isn't a closed form solution for C_t^y as a function of w_t bA_t and the safe interest rate r_{t+1}^s . The only simple case occurs when r_{t+1}^s = g. then no matter what the risky return on capital is, a constant fraction of the wage

$$if\ g=r_{t+1}^{s}\,then\,C_t^y=\frac{W_t}{1+\beta} \quad (11)$$

In this special case, government bonds crowd out private investment one for one with

$$K_{t+1}=\frac{\beta W_t}{1+\beta}-bA_t \quad (12)$$

If r_{t+1}^s < g then government bonds crowd out private investment more than one for one as the expected pension payment causes higher consumption when young.

In this model the labor demand problem is a bit tricky, because the old don't know A_{it} when they chose to hire 1 young worker at wage W_t. At time t the old choose L_{it} (which will be equal to one in equilibrium) to maximize the expected value of utility

$$E\left[\ln\left(bA_t+K_t+K_t^{\alpha}\left(A_{it}L_{it}\right)^{1-\alpha}-W_tL_{it}\right)\right] \quad (13)$$

Equation (14) is a second order Taylor series approximation to (13)

with which implies the first order condition at L_{it} = 1

$$E\frac{(1-\alpha)K_t^{\alpha}A_{it}^{1-\alpha}-W_t}{K_t+bA_t+K_t^{\alpha}A_{it}^{1-\alpha}-W_t}=0 \quad (14)$$

Now define k = K/A and w = W/A and note that $A_{it}^{1-\alpha}/A_t^{1-\alpha}=1+\epsilon_{it}$

So the first order condition becomes

$$E\frac{(1-\alpha)k_t^{\alpha}(1+\epsilon_{it})-w_t}{k_t+b+k_t^{\alpha}(1+\epsilon_{it})-w_t}=0 \quad (15)$$

Which implies $w_t<(1-\alpha)k_t^{\alpha}$

The wage is lower than it would be in the absence of risk, because the old's marginal utility of consumption is higher when A_t is low. Another way of putting this is that the risk reduces labor demand, because higher employment causes the employer to bear more risk.

For small enough σ^2 one can use a second order Taylor series approximation to the utility function to obtain an approximate first order condition at at $L_{it}=1$

$$\frac{(1-\alpha)k_t^{\alpha}-w_t}{k_t+b+k_t^{\alpha}-w_t}-\frac{2(1-\alpha)k_t^{\alpha}(k_t+b+k_t^{\alpha}-w_t)\sigma^2}{2(k_t+b+k_t^{\alpha}-w_t)^2}\approx 0 \quad (16)$$

Which immediately simplifies to

$$w_t\approx(1-\alpha)k_t^{\alpha}(1-\sigma^2) \quad (17)$$

Finally assume balanced growth, so the lower case variables are constant. If $r^s=g$ balanced growth occurs iff

$$k(1+g)=\frac{K_{t+1}}{A_t}=k+\frac{\beta w}{1+\beta}-b \quad (18)$$

Recall that $\frac{\beta w}{1+\beta}-b<k+\frac{\beta(1-\alpha)k^{\alpha}}{1+\beta}-b$ (19)

If $r^s<g$ balanced growth k is reduced due to the effect of pensions on saving (this effect increases in b).

The welfare of an agent born at time t is

$$U_t = \ln\left(\frac{w}{(1+\beta)(1+g)}\right) + \beta E[\ln(b+k+k^{\alpha}(1+\epsilon_{it})-w)] + (t+1)\ln(1+g) \quad (20)$$

An increase in b has three effects in equilibrium. It causes a shift of saving from risky capital to safe bonds. For constant expected return on capital, If r^s and w, agents are indifferent about this shift. If If $r^s<g$, it causes an increase in pension payments which unambiguously cause higher welfare and it causes a reduction of w both by causing reduced k and by causing reduced expected consumption and increased absolute risk aversion of the old. The reduction in w implies a transfer from the young to the old with each agent receiving 1+g units of consumption when old for each unit of consumption lost when young. Since w is not stochastic, this transfer involves no risk. If $r^s<g$ this transfer causes higher welfare for each generation. Agents are indifferent about consuming a bit less when young, saving a bit more and receiving a safe return of r^s g so they strictly prefer the effect of reduced wages which are equivalent to making them save and receive a safe return of g.

This means that, if the safe rate of interest is less than the rate of growth, the economy is dynamically inefficient. This is the case even though total capital income is greater than investment, that is, total consumption is greater than labor income.

The usual simple national income and product accounts based formula for determining whether an economy is dynamically efficient due to Abel, Mankiw, Summers, and Zeckhauser does not work in this case.

More conventional models.

The strong conclusion above follows partly from the strong, and unusual, assumption that uncertainty is resolved after firms chose how many workers to hire. I don't have any particular sense that the standard assumption that employment is chosen after uncertainly is resolved is more reasonable, but I should note that it can make a big difference.

It is still possible, with the extreme one firm each assumption, to get the results exactly as above. I think the best way to do this is to take very literally the assumption of one firm per old person and assume each old person can hire 0, 1 or 2 young workers. Then have the highest possible productivity of the second employee as low is equal to the lowest possible productivity of the first employee. This means that each old person will choose to hire exactly one employee and the wage will be equal to that productivity.

The fact that employment must be an integer implies that, for the firm with the lowest productivity, the worker gets all of the output and profits are zero.

Everything else works out just as in the model above assuming output of a firm with one employee is equal to

$$Y_{it} = K_{it}^{\alpha}(A_{it})^{1-\alpha} \quad (21)$$

And the minimum value of epsilon is given by inequality 22

$$\epsilon \geq (1-\alpha)(1-\sigma^2)$$

Again the result is that public debt crowds out investment and that this causes higher welfare so long as the rate of average technological progress n is greater than the safe interest rate.

However, the assumption of one firm per worker is unrealistic. It might be accepted if it is clearly just a way to model financial frictions, but it is unlikely that anyone will take seriously this model in which the assumption is taken seriously. Rather the standard assumption is that there labor is continuous and there are perfect returns to scale. This may imply workers moonlighting – working for more than one firm, but it is so standard that it is what I assumed in the first model I discussed without even noticing the strangeness of the assumption.

Model III, an actually conventional model with continuous employment and technology resolved before employment is chosen behaves quite differently than the eccentric models above. Firms all pay the same wage w_t so

$$L_{it}=(1-\alpha)^{1/\alpha} K A_{it}^{(1-\alpha)/\alpha} W_t^{-1/\alpha} \quad (22)$$

Average employment equals one so

$$W_t=(1-\alpha) K_t^{\alpha} A_t^{1-\alpha} \left(E\left((1+\epsilon_{it})^{(1-\alpha)/\alpha} \right) \right)^{\alpha} \quad (23)$$

Which is standard except for the extraordinarily ugly term in ϵ

Again public debt will crowd out investment and cause lower wages. Again this amounts to a transfer from the young to the old. However, in this case, the benefit of the reduced wage is greater to the old agents who are already lucky because they have a high A_{it}. This makes the effect of reduced wage a risky transfer just as investment itself is risky and, in fact, with the exact same pattern of risk and return.

Output of firm i is

$$Y_{it}=K_t((1-\alpha) A_{it})^{(1-\alpha)/\alpha} W_t^{(\alpha-1)/\alpha} \quad (24)$$

And, given the conventional labor demand with Cobb-Douglas production, the share of capital is α so the consumption of old person i is

$$C_{it}^{o}=bA_t+K_t\left(1+\alpha\left(K_t((1-\alpha) A_{it})^{(1-\alpha)/\alpha} W_t^{(\alpha-1)/\alpha}\right)\right) \quad (7)$$

This means that the benefit from reduced wages is proportional to total profits. The reduction in wages multiplies the return to investment by a constant greater than one. The distribution around the means of the gain in consumption when old due to an additional unit of saving and due to a unit reduction in wages is identical.

However the average gain to the old due to the reduction in wages is not equal to the average return on capital. Instead, agents gain 1+g units of consumption on average when old in exchange for each unit of reduced wages when young. This means that the wage reduction due to crowding out effect of public debt is like a risky investment with return g. This reduces welfare if the risky return (r^r) that is the expected return on capital, is greater than g.

In the more conventional model, public debt has two contrasting effects when the rate of growth is higher than the safe interest rate and lower than the expected risky return – the pension increases welfare but the reduction in wages reduces welfare.

It is clear that if the rate of growth is equal to the expected return on capital, then public debt causes higher welfare. This means that the standard Abel, Mankiw, Summers, and Zeckhauser calculation is misleading in this case as well.

Here is a very very rough back of the envelope calculation for the USA of the relative magnitudes of the gain to the public sector of a higher debt to GDP ratio and the loss to private agents due to crowding out and reduced wages. First define lower case variables as upper case variables divided by A_t so $k_t = K_t/A_t$ etc,

If debt = bA_t the state can transfer $(g- r^s)\, bA_t$ each period. For my convenience, I will assume that part is given to the young and part is given to the old so the transfer does not affect the amount agents choose to save (if all were given to the old the transfer would lower saving and if all were given to the young it would cause higher saving). This causes an increase in steady state welfare larger than the benefit of transfers only to the young -- there is a term in $(g- r^s)$ which I will ignore.

Logarithmic utility implies a constant saving rate $\frac{K_{t+1}+bA_t}{W_t}=\frac{\beta}{1+\beta}$ so $(1+g)k_{t+1}=\frac{\beta w_t}{(1+\beta)}-b$.

Cobb Douglas production function implies $\frac{\partial w_t}{\partial k_t}=\frac{\alpha w_t}{k_t}$ so $\frac{\partial k_t}{\partial w_t}=\frac{k_t}{\alpha w_t}$ so in balanced growth with constant k

$$(1+g)\frac{dk}{db}=\frac{\beta}{(1+\beta)}\frac{dw}{db}-1=\frac{(1+g)k}{\alpha w}\frac{dw}{db} \quad (8)$$

At b = 0 $\frac{(1+g)k}{w}=\frac{\beta}{1+\beta}$

so at b=0

$$\frac{dw}{db}=\frac{-(1+\beta)\alpha}{\beta(1-\alpha)} \quad (9)$$

This means that a small positive debt to technology ratio b has a first order effect on welfare the same as a windfall of the order of

$$[(g-r^s)+(g-r^r)\frac{(1+\beta)\alpha}{\beta(1-\alpha)}]bA_t \quad (10)$$

The sign depends on the endogenous safe and risky interest rates and the parameters α and β. The assumption α = 1/3 is conventional. The unit of time is one generation, so β is small and the interest rates are very high. Consider a GDP growth rate of 2.5% per year. If a generation is roughly 28 years long, 1+g is roughly 2 . $1+r^r$ is very roughly on the order of 3 (corresponding to a risky rate of roughly 4% per year) to 9 (corresponding to a risky rate of roughly 7% per year) .

$r^r = \alpha k^{1-\alpha}$ So

$$\frac{w}{k} = (1-\alpha)k^{1-\alpha} = (1-\alpha)r^r/\alpha \quad (11)$$

and

$$(1+g)k = 2k = \frac{\beta(1-\alpha)}{(1+\beta)\alpha} r^r k \quad (12)$$

So $\frac{dw}{db} = \frac{r^r}{2}$

For the low risky rate of roughly 4% per year or 200% per generation, this implies that $\frac{dw}{db} \simeq 1$ and the introduction of a small public debt causes increased welfare if $g-r^s > r^r - g$ that is if $r^s < 0\%$ per year. For this unusually low estimate of the risky return, the model implies dynamic inefficiency only if safe real interest rates are negative.

The appropriate safe interest rate is the 30 year inflation protected (TIPS) yield which is currently Yielding 0.83% per year, so the model with employment determined after uncertainty is resolved suggests dynamic efficiency.

Finally, an informal discussion of a genuinely conventional model. There are standard real business cycle models with risky returns on capital in which the risk is aggregate risk. These models have perfectly efficient financial markets. If such assumptions about production are introduced to an OLG model, the model more difficult to analyze because wages, expected returns on capital and aggregate capital are stochastic. However, it is clear that the benefit to the old due to reduced wages is risky. This means that the analysis of the model III should apply to expected values and balanced growth in the conventional OLG model with aggregate risk.

I am not at all willing to continue to try to work out the model (I have tried and not gotten much of anywhere) but I think it is likely that a persistently negative safe real rate of interest is required for dynamic inefficiency.

Both the conventional model with aggregate shocks and the conventional model with idiosyncratic shocks followed by flexible choice of employment include standard but very strong assumptions. First, as stressed above, the models assume full employment so increased consumption due to public debt crowds out investment one for one. If there is any effect of aggregate demand on output at all, the case for increased debt is stronger. Second it is assumed in the conventional models that firms can adjust employment without any cost. This means that, in the models, labor income is always proportional to capital income. In historical data, the share of labor is markedly countercyclical. This suggests that a valid model would assume partial adjustment of employment to shocks which would imply that the critical safe interest rate which implies dynamic inefficiency is positive but smaller than the growth rate. Finally, all the models assume perfect competition in product markets. If firms have market power, then rents are treated as capital income in national accounts and the true marginal product of capital would be less than the ratio of capital income to capital. The risky rate of interest is the expected marginal product of capital. If it is significantly over estimated because of imperfect competition the US economy may be dynamically inefficient even though safe real interest rates are positive.

The fact that interest rates are close to the critical values calculated with models based on extreme assumptions which tend to imply dynamic efficiency convinces me that it is reasonably likely that increased public debt would cause increased welfare.

* I want to thank Nick Rowe for a very useful comment.