Lecture Notes: Econ 101B: August 29-31 2006

Introduction to the Theory of Economic Growth

Questions

Why is the world so much richer today than it was fifty or a hundred years ago? What are the prospects for increasing riches in the future? And why is the world today so unequal as we look across countries?

Macroeconomists study these questions in their standard way. Ruthless simplification, followed by specifying aggregate behavioral relationships and equilibrium conditions, and then analyzing the consequences of their assumptions and trying to map their conclusions onto the world.

Economists make an intellectual bet: **that an exact solution to a grossly oversimplified model that approximates the most important features of the world will be a reasonably good approximation to what is actually going on in the world.**

Basics: The Production Function

So, begin by assuming that four things are important: labor L, capital K, technology and organization E, and *diminishing returns to scale*. We begin with a production function:

$$Y = K^{\alpha}(EL)^{1-\alpha}$$

with α a parameter between 0 and 1, governing diminishing returns to scale.

We are going to want to work with rates of change, and with log rates of change—which are the same thing as proportional rates of growth:

$$\frac{d\ln(x)}{dt} = \frac{1}{x}\frac{dx}{dt}$$

So let's take logs and then the time derivative of our production function:

$$\ln(Y) = \alpha\ln(K) + (1-\alpha)\ln(L) + (1-\alpha)\ln(E)$$

$$\frac{d\ln(Y)}{dt} = \alpha\frac{d\ln(K)}{dt} + (1-\alpha)\frac{d\ln(L)}{dt} + (1-\alpha)\frac{d\ln(E)}{dt}$$
$$\frac{1}{Y}\frac{dY}{dt} = \alpha\left(\frac{1}{K}\frac{dK}{dt}\right) + (1-\alpha)\left(\frac{1}{L}\frac{dL}{dt}\right) + (1-\alpha)\left(\frac{1}{E}\frac{dE}{dt}\right)$$

Basics: The Capital-Output Ratio

We are also going to want to think about the capital-output ratio κ:

$$\kappa = \frac{K}{Y}$$
$$\ln(\kappa) = \ln(K) - \ln(Y)$$

and:

$$\frac{d\ln(\kappa)}{dt} = \frac{d\ln(K)}{dt} - \frac{d\ln(Y)}{dt} = (1-\alpha)\left[\frac{d\ln(K)}{dt} - \frac{d\ln(L)}{dt} - \frac{d\ln(E)}{dt}\right]$$

Behavioral Relationships

Let us start with some very simple behavioral relationships:

Households do their things: The labor force L grows at a constant proportional rate n:

$$\frac{d\ln(L)}{dt} = n \quad \Rightarrow \quad L_t = L_0 e^{nt}$$

Scientists and entrepreneurs do their thing The efficiency of labor E grows at a constant proportional rate g:

$$\frac{d\ln(E)}{dt} = g \quad \Rightarrow \quad E_t = E_0 e^{gt}$$

Executives and investors do their thing. A fraction δ of the capital stock wears out each year, but society saves and invests a fraction s of output Y:

$$\frac{d\ln(K)}{dt} = s\left(\frac{Y}{K}\right) - \delta = \frac{s}{\kappa} - \delta \quad \Rightarrow \quad K_t = ??? \quad \kappa_t = ???$$

Everybody keeps on doing their thing from time zero to the end of time. What happens?

Imposing an Equilibrium Condition

Let's look for a situation in which the capital-labor ratio is stable: that will be our equilibrium condition:

$$0 = \frac{d\ln(\kappa)}{dt} = (1-\alpha)\left[\frac{d\ln(K)}{dt} - \frac{d\ln(L)}{dt} - \frac{d\ln(E)}{dt}\right]$$

Substitute in:

$$0 = (1-\alpha)\left[\frac{s}{\kappa} - \delta - n - g\right]$$

And solve for the equilibrium capital-output ratio κ*--the capital-output ratio where there is neither upward nor downward pressure on it over time:

$$\kappa^* = \frac{s}{n + g + \delta}$$

If the economy gets to that capital-output ratio, it will stay there, in which case the capital stock will be, at some arbitrary time t:

$$K_t = \kappa^* Y_t$$

and output and output per worker will be:

$$Y_t = (\kappa^* Y_t)^\alpha (E_t L_t)^{1-\alpha}$$
$$Y_t^{1-\alpha} = (\kappa^*)^\alpha (E_t L_t)^{1-\alpha}$$
$$Y_t = (\kappa^*)^{\frac{\alpha}{1-\alpha}} E_t L_t$$
$$\frac{Y_t}{L_t} = (\kappa^*)^{\frac{\alpha}{1-\alpha}} E_t$$
$$\frac{Y_t}{L_t} = \left(\frac{s}{n + g + \delta}\right)^{\frac{\alpha}{1-\alpha}} E_t$$

And we will be done with our analysis—if we can be confident that the capital-output ratio will be at its equilibrium, its steady-state balanced-growth, value. Will it?

Sticking Our Toe into Differential Equations

Let us go back to:

$$\frac{d\ln(\kappa)}{dt} = (1-\alpha)\left[\frac{d\ln(K)}{dt} - \frac{d\ln(L)}{dt} - \frac{d\ln(E)}{dt}\right]$$

and substitute in:

$$\frac{d\ln(\kappa)}{dt} = (1-\alpha)\left[\frac{s}{\kappa} - \delta - n - g\right]$$

$$\frac{1}{\kappa}\frac{d\kappa}{dt} = (1-\alpha)\left[\frac{s}{\kappa} - \delta - n - g\right]$$

$$\frac{d\kappa}{dt} = (1-\alpha)\left[s - (\delta + n + g)\kappa\right]$$

Now remember our expression for κ*, write:

$$\kappa = (\kappa - \kappa^*) + \kappa^*$$

and substitute in:

$$\frac{d\kappa}{dt} = (1-\alpha)\left[s - (\delta + n + g)(\kappa^* + (\kappa - \kappa^*))\right]$$

$$\frac{d\kappa}{dt} = (1-\alpha)\left[s - (\delta + n + g)\kappa^* - (\delta + n + g)(\kappa - \kappa^*)\right]$$

$$\frac{d\kappa}{dt} = (1-\alpha)\left[s - (\delta + n + g)\frac{s}{\delta + n + g} - (\delta + n + g)(\kappa - \kappa^*)\right]$$

$$\frac{d\kappa}{dt} = -(1-\alpha)(\delta + n + g)(\kappa - \kappa^*)$$

This last is a very special equation, with solution:

$$\kappa_t = \kappa^* + (\kappa_0 - \kappa^*)e^{-(1-\alpha)(n+g+\delta)}$$

So yes, this economy will "eventually" get the capital-output ratio to its steady-state balanced growth value. How eventually? It depends on how fast the dying exponential term in the equation above takes to become really small.

We Now Have a Cookbook

To analyze where an economy is and where it is going:

- Look at its parameters—n, g, δ, s, and α.
- Calculate its steady-state capital-output ratio κ^*.
- Look at its initial capital-output ratio κ_0 and calculate its current capital-output ratio κ_t using:
  $\kappa_t = \kappa^* + (\kappa_0 - \kappa^*)e^{-(1-\alpha)(n+g+\delta)}$
- Calculate output-per-worker and output, using:
  $$Y_t = (\kappa_t)^{\frac{\alpha}{1-\alpha}} E_t L_t$$
  $$\frac{Y_t}{L_t} = (\kappa_t)^{\frac{\alpha}{1-\alpha}} E_t$$

To analyze the effects of a change in the economy:

- Start with its position on its steady-state balanced-growth path for the old, unchanged parameter values.
- Calculate the new steady-state balanced-growth path for the new, changed parameter values.
- Watch the economy transit from its old to its new steady-state balanced-growth path.

We Now Have a Base Camp

We can now explore in all directions: modifying our model to add one effect or another, calculating the consequences of complicating the model in that way, and then returning to our base case to conduct another exploration.

Explorations we may do:

- Model with natural resources
- Model with resource-augmenting (rather than labor-augmenting) productivity growth
- Model with Malthusian population dynamics
- Model with demographic transition
- Model with relative price structure-induced poverty traps
- Model where two heads are better than one
- Model with short-lived computer capital
- Et cetera…